Insight: Cambodia Journal of Basic and Applied Research, Volume 3 No. 2 (2021)

Spectral Analysis of Banach Spaces and their Application to Age-Structured Equations

MEAS Len*
Department of Mathematics, Royal University of Phnom Penh, Russian Federation Boulevard, Toul Kork, Phnom Penh, Cambodia
*Corresponding Author: MEAS Len (meas.len@rupp.edu.kh)

To cite this article: Meas, L. (2021) Spectral Analysis of Banach Spaces and their Application to Age-Structured Equations. *Cambodia Journal of Basic and Applied Research (CJBAR), 3(1)*, 32–69.

សង្ខិត្តន័យ

ក្នុងអត្ថបទនេះ យើងនឹងប្រើទ្រឹស្តីបទ Floquet សម្រាប់ការវិជ្ជមាននិងមានខួបលើលំហ Banach ដើម្បីសិក្សាភាពមានចម្លើយនិងភាពមានចម្លើយតែមួយគត់នៃបញ្ហាតម្លៃផ្ទាល់ Floquet និងបញ្ហា adjoint របស់វា។ នៅផ្នែកចុងក្រោយ យើងនឹងប្រើទ្រឹស្តីបទនេះទៅលើសមីការសម្ព័ន្ធអាយុ ដែលមានមេគុណវិជ្ជមាននិងមានខួប ដើម្បីសិក្សាស្វ័យគុណ Floquet ដែលជារង្វាស់នៃអាត្រាកំណើន។ ស្របពេលជាមួយគ្នានេះ យើងនឹងទទួលបាននូវការថយចុះបែបអ៊ិចស្ប៉ូណង់ស្យែលនិងអាស៊ីមតូតរយៈពេលវែងតាមវិធីសាស្ត្រអង់ត្រូពី។

Abstract

In this paper, Floquet's theory will be applied to a positive periodic operator on a Banach space to show the existence and uniqueness of a solution to Floquet eigenvalue problems and their adjoints. Then, the theory will be applied to an age-structured equation with positive and periodic coefficients to study a Floquet exponent, which measures the growth rate of a population. At the same time, exponential and long-run asymptotic decay will be derived using the entropy method.

Keywords: Eigenvalue problem, age-structured equation, Floquet theory

Insight: Cambodia Journal of Basic and Applied Research, Volume 3 No. 2 (2021)

Introduction

When modelling population dynamics, the first step is to identify significant variables that enable the division of a population into homogeneous subgroups. This is used to describe the dynamics of the interaction between these groups. Age is one of the most natural and significant parameters for structuring a population. Many internal variables are dependent on age. For example, age differences may be associated with different reproductive and survival abilities.

A model for age-structured populations (McKenDrick, 1926; von Foerster, 1959) was designed to study disease transmission in populations. Often diseases have different infection and mortality rates for different age groups (Anderson & May, 1991). For instance, chickenpox or measles are spread mainly via contact between two members of a population of a similar age. In models of disease transmission, an age-structured equation is useful as it allows the ages of different members of a population to be accounted for when determining variables such as contact rates.

Iannelli & Milner (2017) defined the evolution of a population over time using an age density function known as the McKendrick equation. There are several reasons for introducing time dependence between the coefficients of this equation. A common rationale is to represent seasonality. Another is related to modelling cell division in cancer treatments, such as resonance and chrono-therapy, which are based on modelling Circadian rhythms (Clairambault, Michel & Perthame, 2016).

In this paper, an age-structured model is analyzed for the periodic death and birth rate of a population over time. It uses a partial differential evolution equation (Iannelli & Milner, 2017) that models the dynamic nature of the population density $n(t,x)$ of individuals aged $x > 0$ at a time $t \in (0,\infty)$ with age-dependent birth and death rates. The age-structured equation has the following form

$$\begin{cases} \dfrac{\partial}{\partial t}n(t,x) + \dfrac{\partial}{\partial x}n(t,x) + d(t,x)n(t,x) = 0, \ \forall t \geq 0, x \geq 0 \\ n(t, x = 0) = \displaystyle\int_0^\infty B(t,x)n(t,x)dx \\ n(t = 0, x) = n^0(x), \end{cases}$$

where $d(t,x)\, and\, B(t,x)$ represent the death and birth rate of a population, respectively, as periodic functions with a period T.

This work derives a solution for the age-structured equation and its long-run asymptotic exponential decay, as well as a proof of its uniqueness.

Methodology

An age-structured model based on a partial differential evolution equation has been used to predict population density dynamics. More specifically, this is an eigenvalue problem that poses some fundamental questions about the existence and uniqueness of these equations. To answer these questions, Floquet's theory is applied to a Banach space. This is an extension of applying Floquet's theory to a matrix (Brown, Easthem & Schmidt, 2013) to prove the existence of a Floquet exponent. Then, the long run asymptotic exponential decay of the solution of the age-structured equation is proven via the entropy method (Perthame, 2007; Michel, Mischler & Perthame, 2004; 2005).

Results and Findings

This work comprises two parts. The first deals with the extension of Floquet's theory for any positive periodic matrix to any positive periodic operator on a Banach space. More specifically, a linear differential equation of the form

$$\frac{d}{dt}X(t) = A(t)X(t),$$

where $t \in R_+, X(t)$ is a vector on a Banach space E and $A(t)$ is a periodic continuous operator with period T on E.

The existence and uniqueness of the Floquet exponent λ_{per} and the positive and T -periodic $N(t), \phi(t)$ will be proven for the following equations

$$\frac{dN(t)}{dt} = A(t)N(t) - \lambda_{per}N(t) \text{ and } -\frac{d\phi(t)}{dt} = A^*(t)\phi(t) - \lambda_{per}\phi(t).$$

The second part of this paper applies these results to an age-structured equation. Partial differential evolution equations using coefficients that are periodic functions of time are used to model population density dynamics. The existence and uniqueness of $(\lambda_{\text{per,}}\, N, \phi)$ will be proven for the following age-structured equation:

$$\begin{cases} \dfrac{\partial}{\partial t}n(t,x) + \dfrac{\partial}{\partial x}n(t,x) + d(t,x)n(t,x) = 0, \ \forall t \geq 0, x \geq 0 \\ n(t, x=0) = \displaystyle\int_0^\infty B(t,x)n(t,x)dx \\ n(t=0,x) = n^0(x). \end{cases}$$

The associated Floquet eigenvalue problem of the age-structured equation above is given by:

$$\begin{cases} \frac{\partial}{\partial t}N(t,x) + \frac{\partial}{\partial x}N(t,x) + \left(\lambda_{per} + d(t,x)\right)N(t,x) = 0, \ \forall t \geq 0, x \geq 0 \\ N(t, x = 0) = \int_0^\infty B(t,x)N(t,x)dx \\ N(t,x) > 0, T - periodic, \int_0^T \int_0^\infty N(t,x)dxdt = 1, \end{cases}$$

And it's adjoint eigenvalue problem is given by

$$\begin{cases} -\frac{\partial}{\partial t}\phi(t,x) - \frac{\partial}{\partial x}\phi(t,x) + \left(\lambda_{per} + d(t,x)\right)\phi(t,x) = B(t,x)\phi(t,0), \\ \hfill \forall t \geq 0, x \geq 0 \\ \phi(t,x) > 0, T - periodic, \int_0^\infty N(t,x)\phi(t,x)dx = 1. \end{cases}$$

The long-run asymptotic exponential decay of this equation is derived as follows:

$$\int_0^\infty \left|n(t,x)e^{-\lambda_{per}t} - \rho N(t,x)\right|\phi(t,x)dx \\ \leq e^{-\alpha t}\int_0^\infty |n^0(x) - \rho N(0,x)|\phi(0,x)dx.$$

where $\rho = \int_0^\infty n^0(x)\phi(0,x)dx$ and its long-run asymptotic behaviour via the entropy method

$$\int_0^\infty \left|n(t,x)e^{-\lambda_{per}t} - \rho N(t,x)\right|\phi(t,x)dx \to \ 0 \ as \ t \to \infty.$$

Floquet's Theory

Floquet theory for matrix. The following homogeneous linear periodic system

$$\frac{d}{dt}X(t) = A(t)X(t), \tag{1.1}$$

where $X \in R^d$ and $A(t)$ is a continuous $d \times d$ real matrix-valued function in t, and $A(t+T) = A(t), for\ some\ T > 0$. A unique solution exists for Equation 1.1 for the initial condition $X(t_0) = x_0$. This solution satisfies $X(t) = \Phi(t, t_0)x_0$, where $\Phi(t, t_0)$, is known as a principal fundamental matrix solution and is a solution to the matrix initial value problem

$$\frac{d}{dt}\Phi(t, t_0) = A(t)\Phi(t, t_0),\ \ \Phi(t_0, t_0) = I.$$

1. As this solution is unique,

$$\Phi(t, r) = \Phi(t, s)\Phi(s, r),\ \ \forall r < s < t$$

2. $\Phi(t, t_0) = \Phi(t_0, t)^{-1}$.

Thus, it may be observed that

$$\begin{cases} \dfrac{d}{dt}\Phi(t+T, t_0+T) = A(t+T)\Phi(t+T, t_0+T) = A(t)\Phi(t+T, t_0+T) \\ \Phi(t_0+T, t_0+T) = I. \end{cases}$$

Again, due to the uniqueness of the solution, $\Phi(t+T, t_0+T) = \Phi(t, t_0)$.

Now, it may be denoted that,

$$M(t_0) \coloneqq \Phi(t_0+T, t_0) \text{ and } M \coloneqq \Phi(T, 0).$$

Then, it follows that

$$\begin{aligned} M(t_1) = \Phi(t_1+T, t_1) &= \Phi(t_1+T, t_0+T)\Phi(t_0+T, t_0)\Phi(t_0, t_1) \\ &= \Phi(t_1, t_0)M(t_0)\Phi(t_1, t_0)^{-1}. \end{aligned}$$

This means that $M(t_0)$ and $M(t_1)$ are similar for $t_0 < t_1$ and thus have the same eigenvalues. That is, the eigenvalues of $M(t_0)$ are independent of t_0. Thus, the eigenvalues of $M = \Phi(T, 0)$, also known as a monodromy matrix, are of interest to this study. As $det(M) \neq 0$, a constant matrix B exists, whereby $M = e^{TB}$.

Definition 1.1. The eigenvalues ρ_j of M are called Floquet multipliers. The complex eigenvalues λ_j of B are called Floquet exponents and are related by the equation $\rho_j = e^{\lambda_j T}$.

Theorem 1.2 (Floquet). If M is a monodromy matrix for a T-periodic linear system (Equation 1.1). Then, there is an invertible periodic matrix $P(t)$ and a constant matrix B such that

$\Phi(t,0) = P(t)e^{tB}\ for\ any\ t > 0.$

Proof. If $\Psi(t) \coloneqq \Phi(t+T,0)$ then the following initial value problem is satisfied

$$\begin{cases} \dfrac{d}{dt}\Psi(t) = A(t+T)\Psi(t) = A(t)\Psi(t), \\ \Psi(0) = M \end{cases}$$

since $A(t)$ is T-periodic, a unique solution is given by $\Psi(t) = \Phi(t,0)M$.

Thus,

$$\Phi(t+T,0) = \Phi(t,0)M = \Phi(t,0)e^{TB}.$$

By taking $P(t) \coloneqq \Phi(t,0)e^{-tB}$,

$$P(t+T) = \Phi(t+T,0)e^{-(t+T)B} = \Phi(t,0)e^{TB}e^{-(t+T)B} = P(t)$$

and $P(0) = I$.

Theorem 1.3. There exists a real $2T$-periodic matrix $Q(t)$ and a real matrix R such that

$\Phi(t,0) = Q(t)e^{tR}.$

Proof. Since $det(M) \neq 0$, there exists a real matrix R such that $M^2 = e^{2TR}$.

Thus, it may be defined that $(t) \coloneqq \Phi(t,0)e^{-tR}$.

Then, it follows that

$$Q(t+2T) = \Phi(t+2T,0)e^{-2TR}e^{tR} = \Phi(t+T,0)Me^{-2TR}e^{tR}$$
$$= \Phi(t,0)M^2e^{-2TR}e^{tR} = \Phi(t,0)M^2M^{-2}e^{tR} = Q(t).$$

Therefore, Q is $2T$-periodic.

Theorem 1.4. If ρ_j is a characteristic multiplier and λ_j is a corresponding characteristic exponent so that $\rho_j = e^{\lambda_j T}$; then a solution $X(t)$ exists for Equation 1.1, such that

1. $X(t+T) = \rho_j X(t)$
2. $X(t) = N(t)e^{\lambda_j t}$, where $N: R_+ \to R^d$ is a T- periodic function.

Proof. If ρ_j is an eigenvalue of M, then $v_j \neq 0$ and $Mv_j = \rho_j v_j$. Thus, if $X(t) = \Phi(t,t_0)v_j$, then $X(t)$ satisfies the initial value problem

$$\begin{cases} \dfrac{d}{dt}X(t) = A(t)X(t), \\ X(t_0) = v_j \,. \end{cases}$$

It follows that

$$X(t+T) = \Phi(t+T,t_0)v_j = \Phi(t,t_0)Mv_j = \rho_j\Phi(t,t_0)v_j = \rho_j X(t).$$

Furthermore, by taking $N(t) = X(t)e^{-\lambda_j t}$ and using the fact that $\rho_j = e^{\lambda_j T}$, it may be stated that

$$N(t+T) = X(t+T)e^{-\lambda_j(t+T)} = \rho_j X(t)e^{-\lambda_j t}e^{-\lambda_j T} = X(t)e^{-\lambda_j t} = N(t).$$

So, when $A(t)$ is a positive T-periodic function, and if $\int_0^T A(t)dt$ is positive and irreducible, then matrix B in **Theorem 1.2** is also positive and irreducible, leading to the following corollary.

Corollary 1.5. There exists a Floquet exponent $\lambda_{\text{per}} > 0$ and a T-periodic $N(t) > 0$ satisfying

$$\frac{dN(t)}{dt} = A(t)N(t) - \lambda_{per}N(t).$$

Proof. Since B is positive and irreducible, so is M. Thus, by the Perron-Frobenius theorem, an eigenvalue $\lambda > 0$ of B exists and is associated with a positive eigenvector. If it is taken that $\lambda_{\text{per}} = \lambda > 0$. Then $e^{\lambda_{per}T}$ is a positive eigenvalue of M associated with a positive eigenvector v.

Then it may be defined that

$$N(t) = X(t)e^{-\lambda_{per}t},$$

where $X(t) = \Phi(t, t_0)v$ and the desired result follows from **Theorem 1.4.**

Next, the adjoint linear periodic system can be considered

$$\frac{d}{dt}Z(t) = -A^*(t)Z(t). \tag{1.2}$$

Given $Z(t_0) = z_0$, it has a unique solution $Z(t) = \Psi(t_0, t)z_0$, where $\Psi(t_0, t)$ is defined as the matrix solution to

$$\frac{d}{dt}\Psi(t_0, t) = -A^*(t)\Psi(t_0, t),\ \Psi(t_0, t_0) = I.$$

The monodromy matrix M^* then may be defined as follows:

$$M^* \coloneqq \Psi(0, T) = \left(\Phi^*(0, T)\right)^{-1} = \left(\Phi(T, 0)\right)^{-1} = M^{-1}.$$

Theorem 1.6. If ρ_j be a characteristic multiplier and λ_j is a corresponding characteristic exponent so that $\rho_j = e^{\lambda_j T}$. Then there exists a solution $Z(t)$ to Equation 1.2 such that

1. $Z(t + T) = \frac{1}{\rho_j}Z(t).$
2. $Z(t) = \phi(t)e^{-\lambda_j t}$ for some T-periodic function $t \mapsto \phi(t) \in R^d$.

Proof. If ρ_j be an eigenvalue of M, then, $v_j \neq 0$ such that $Mv_j = \rho_j v_j$. Let $Z(t) = \Psi(t_0, t)v_j$. Thus $Z(t)$ satisfies the following initial value problem

$$\begin{cases} \dfrac{d}{dt}Z(t) = -A^*(t)Z(t), \\ Z(t_0) = v_j \end{cases}$$

Then,

$$Z(t+T) = \Psi(t_0, t+T)v_j = \Psi(t_0, t)M^{-1}\, v_j = \rho_j^{-1}\Psi(t_0, t)v_j = \frac{1}{\rho_j}Z(t).$$

And it may be defined that $\phi(t) = Z(t)e^{\lambda_j t}$. Thus if $\rho_j = e^{\lambda_j T}$, it follows that

$$\phi(t+T) = Z(t+T)e^{\lambda_j(t+T)} = \rho_j^{-1}Z(t)e^{\lambda_j t}e^{\lambda_j T} = Z(t)e^{\lambda_j t} = \phi(t).$$

Corollary 1.7. Under the assumption that $A(t)$ is positive and T-periodic, a Floquet exponent $\lambda_{\mathrm{per}} > 0$ and a T-periodic $\phi(t) > 0$ exists satisfying

$$-\frac{d\phi(t)}{dt} = A^*(t)\phi(t) - \lambda_{per}\phi(t).$$

Proof. Since B is positive and irreducible, so is M. Therefore, by the Perron-Frobenius theorem there exists an eigenvalue $\lambda > 0$ of B with an associated positive eigenvector. If it is taken that $\lambda_{\mathrm{per}} = \lambda > 0$, then $e^{\lambda_{per}T}$ is a positive eigenvalue of M associated with a positive eigenvector v. Thus it may be defined that

$$\phi(t) = Z(t)\mathrm{e}^{\lambda_{\mathrm{per}}\mathrm{t}},$$

where $Z(t) = \Psi(t_0, t)v$ and the desired result follows from **Theorem 1.6**.

Floquet theory on Banach space

If a linear periodic system on a Banach space E is considered,

$$\frac{d}{dt}X(t) = A(t)X(t). \tag{1.3}$$

where $X \in E$ and $A(t)$ is a continuous operator-valued T-periodic.

Then is a unique solution to Equation 1.3 together with the initial value $X(t_0) = x_0 \in E$. This solution is given by $X(t) = U(t, t_0)x_0$, where $U(t, t_0)$ is a linear and bounded operator on E and satisfies the following properties:

1. $\frac{d}{dt}U(t, t_0) = A(t)U(t, t_0),\ U(t_0, t_0) = I$
2. $U(t, r) = U(t, s)U(s, r),\ for\ any\ r \leq s \leq t$
3. $U(t + T, t_0 + T) = U(t, t_0)$.

If an operator-valued function is specified as $M(t_0) \coloneqq U(t_0 + T, t_0)$ and a monodromy operator as $M \coloneqq U(T, 0)$, then the following can be denoted:

Definition 1.8. μ is an eigenvalue of M if $\mu \in \sigma_p(M)$, thus, there is a non-zero vector v of E such that $Mv = \mu v$. This vector is an eigenvector corresponding to the eigenvalue μ of M.

Theorem 1.9. The following results hold:

1. $M(t_0 + T) = M(t_0)$
2. $\sigma_p\big(M(t_0)\big) = \sigma_p(M)$ *where* $\sigma_p(M) = \{\mu : \mu I - M$ *is not one-to-one*$\}$.

Proof.

1. By the definition of M, it follows that
$$M(t_0 + T) = U(t_0 + T + T, t_0 + T) = U(t_0 + T, t_0) = M(t_0).$$
2. It can be proven that for $t_0 < t_1, \sigma_p\big(M(t_0)\big) = \sigma_p\big(M(t_1)\big)$. For instance, if $\mu \in \sigma_p\big(M(t_0)\big)$, then $0 \neq v \in E$ exists, such that $M(t_0)v = \mu v$. Then, if $w \coloneqq U(t_1, t_0)v$, it follows that
$$\begin{aligned} M(t_1)w = U(t_1 + T, t_1)w &= U(t_1 + T, t_1)U(t_1, t_0)v \\ &= U(t_1 + T, t_0)v = U(t_1 + T, t_0 + T)U(t_0 + T, t_0) \\ &= U(t_1, t_0)M(t_0)v = \mu U(t_1, t_0)v = \mu w. \end{aligned}$$

That is, $\mu \in \sigma_p\big(M(t_1)\big)$. This means that $\sigma_p\big(M(t_0)\big) \subseteq \sigma_p\big(M(t_1)\big)$.

Conversely, it may be said that if $n_0 \in N$ large enough, so that $n_0 T + t_0 > t_1$, then $\sigma_p\big(M(t_1)\big) \subseteq \sigma_p\big(M(n_0 T + t_0)\big)$. Finally, since $M(t_0)$ is T-periodic, then

$$\sigma_p\big(M(t_0)\big) \subseteq \sigma_p\big(M(t_1)\big) \subseteq \sigma_p\big(M(n_0 T + t_0)\big) = \sigma_p\big(M(t_0)\big).$$

Theorem 1.10. If $\mu = e^{\lambda T}$, then the following are equivalent.

1. μ is an eigenvalue of M
2. A T-periodic function $t \mapsto N(t) \in E$ exists, where $X(t) = N(t)e^{\lambda t}$ is a solution to Equation 1.3 with an initial value $X(t_0) = x_0$.

Proof. In the theorem above, (1) implies (2). Following the same process as the proof for this matrix, (2) also implies (1). Then, if $y_0 \coloneqq U(0, t_0)x_0$, it follows that

$$\begin{aligned} N(t+T) &= X(t+T)e^{-\lambda(t+T)} \\ &= U(t+T, t_0)x_0 e^{-\lambda(t+T)} \\ &= U(t+T, T)U(T, 0)y_0 e^{-\lambda(t+T)} \\ &= U(t, 0)U(T, 0)y_0 e^{-\lambda(t+T)}. \end{aligned}$$

Or alternatively,

$$\begin{aligned} N(t+T) &= N(t) \\ &= X(t)e^{-\lambda t} \\ &= U(t, t_0)x_0 e^{-\lambda t} \\ &= U(t, 0)y_0 e^{-\lambda t}. \end{aligned}$$

Therefore,

$$U(t, 0)U(T, 0)y_0 = e^{\lambda T} U(t, 0)y_0.$$

By taking $t = 0$,

$$My_0 = U(T,0)y_0 = e^{\lambda T}y_0$$

It follows that $e^{\lambda T}$ is an eigenvalue of M.

Corollary 1.11. If additionally, an operator $U(t,0)$ is compact and strictly positive on a Banach lattice, then a Floquet exponent $\lambda_{\text{per}} > 0$ and a T-periodic $N(t) > 0$ exists, satisfying

$$\frac{dN(t)}{dt} = A(t)N(t) - \lambda_{per}N(t). \tag{1.4}$$

Proof. Since $M = U(T,0)$ is compact and strictly positive, the Krein-Rutman theorem demonstrates that there is a simple eigenvalue $\mu > 0$ with an associated eigenvector $N_0 > 0$. Taking $\lambda_{per} > 0$, such that $\mu = e^{\lambda_{per}T}$ and defining $N(t) = X(t)e^{-\lambda_{per}t} > 0$, such that $X(t)$ is defined as in **Theorem 1.10**, the desired result is yielded.

Corollary 1.12. (Uniqueness). There is a unique solution (up to a multiplicative constant) to the Floquet eigenvalue problem in Equation 1.4.

Proof. If another positive T-periodic solution $M(t)$ exists for Equation 1.4, it can be proven that $N(t) = cM(t)$ as follows

$$\frac{d}{dt}\left(e^{\lambda_{per}t}M(t)\right) = A(t)e^{\lambda_{per}t}M(t).$$

The uniqueness of the solution to Equation 1.3 with the initial value gives

$$M(t) = e^{-\lambda_{per}t}U(t,t_0)x_0.$$

Taking $t = T$,

$$\begin{aligned} M(0) = M(T) = e^{-\lambda_{per}T}U(T,t_0)x_0 &= e^{-\lambda_{per}T}U(T,0)U(0,t_0)x_0 \\ &= e^{-\lambda_{per}T}U(T,0)M(0). \end{aligned}$$

Since $M(0) > 0$, $e^{\lambda_{\text{per}}T}$ is an eigenvalue of $M = U(T,0)$ with an associated eigenvector $M(0)$, then $e^{\lambda_{\text{per}}T}$ is a simple eigenvalue of $U(T,0)$ with an associated eigenvector N_0. Hence $M(0) = cN_0$. Thus,

$$\begin{aligned} M(t) &= e^{-\lambda_{per}t}U(t,t_0)x_0 \\ &= e^{-\lambda_{per}t}U(t,0)M(0) \\ &= e^{-\lambda_{per}t}U(t,0)cN_0 \\ &= cN(t). \end{aligned}$$

For the adjoint linear periodic system

$$\frac{d}{dt}Z(t) = -A^*(t)Z(t), \tag{1.5}$$

where $A(t)$ is a continuous T-periodic linear operator-valued function, the following can be denoted.

Theorem 1.13. If $\mu = e^{\lambda T}$, then the following are equivalent.

1. µ is an eigenvalue of M
2. A T-periodic function $t \mapsto \phi(t) \in E$ such that $Z(t) = \phi(t)e^{-\lambda t}$ where $Z(t)$ is the solution of (2.5) with an initial value $Z(t_0) = z_0$ exists.

Proof. In the theorem above, (1) implies (2). Following the same process as the proof for this matrix, (2) also implies (1). Thus

$$\begin{aligned} \phi(t+T) &= Z(t+T)e^{\lambda(t+T)} \\ &= \Psi(t_0, t+T)z_0\, e^{\lambda(t+T)} \\ &= \Psi(t_0,t)\Psi(0,T)z_0\, e^{\lambda(t+T)}. \end{aligned}$$

Or alternatively,

$$\begin{aligned} \phi(t+T) &= \phi(t) \\ &= Z(t)e^{\lambda t} \\ &= \Psi(t_0,t)z_0\, e^{\lambda t}. \end{aligned}$$

Thus

$$\Psi(t_0, t)\Psi(0, T)z_0 = \Psi(t_0, t)z_0\, e^{-\lambda T}$$

Taking $t = t_0$,

$$M^{-1}z_0 = \Psi(0, T)z_0 = e^{-\lambda T}z_0\,.$$

That is, $e^{\lambda T}$ is an eigenvalue of M.

Corollary 1.14. In addition, if the operator $U(t, 0)$ is compact and strictly positive on a Banach lattice, a Floquet exponent $\lambda_{per} > 0$ and a T-periodic $\phi(t) > 0$ exists satisfying

$$-\frac{d\phi(t)}{dt} = A^*(t)\phi(t) - \lambda_{per}\phi(t).$$

Proof. Since $M = U(T, 0)$ is compact and strictly positive, the Krein-Rutman theorem demonstrates that there is a simple eigenvalue $\mu > 0$ and associated eigenvector $\phi_0 > 0$. If it is taken that $\lambda_{per} > 0$, such that $\mu = e^{\lambda_{per}T}$ and $\phi(t) = Z(t)e^{\lambda_{per}t} > 0$, where $Z(t)$ is defined as in **Theorem 1.13**, the desired result is obtained.

General relative entropy

Definition 2.1. (Perthame, 2007, p. 165) If H is a real-valued convex function, then the general relative entropy (GRE) may be defined as

$$\sum_{i=1}^{d} H\left(\frac{X_i(t)}{N_i(t)}\right) N_i(t)\phi_i(t),$$

where $X_i\,(t), N_i(t)\ and\ \phi_i\,(t)$ satisfy Equations 2.1, 2.2 and 2.3, respectively.

The uniqueness of the solution to the Floquet eigenvalue problem

If $A(t) = \left(a_{ij}(t)\right) \geq 0$ for $1 \leq i, j \leq d$, then $X_i(t) > 0, N_i(t) > 0, \phi_i(t) > 0$ and it can also be proven that a unique λ_{per} exists with a maximal real part such that

$$\frac{dX_i(t)}{dt} = \sum_j a_{ij}(t)X_j(t) - \lambda_{per}X_i(t) \tag{2.1}$$

$$\frac{dN_i(t)}{dt} = \sum_j a_{ij}(t)N_j(t) - \lambda_{per}N_i(t) \tag{2.2}$$

$$-\frac{d\phi_i(t)}{dt} = \sum_j a_{ji}(t)\phi_j(t) - \lambda_{per}\phi_i(t). \tag{2.3}$$

Theorem 2.2. If $A(t)$ be a T-periodic matrix with $\left(\lambda_{per}, N, \phi\right)$ defined as above, then for any positive initial conditions, for any solution of Equation 2.1, and any positive convex function H, then

$$\forall t \geq 0, \ \frac{d}{dt}\sum_{i=1}^{d} H\left(\frac{X_i(t)}{N_i(t)}\right)N_i(t)\phi_i(t) = -D_H(X)(t),$$

where

$$D_H(X)(t) = \sum_{i,j} a_{ij}(t)N_j(t)\phi_i(t)\left[H'\left(\frac{X_i(t)}{N_i(t)}\right)\left(\frac{X_i(t)}{N_i(t)} - \frac{X_j(t)}{N_j(t)}\right) + H\left(\frac{X_j(t)}{N_j(t)}\right) - H\left(\frac{X_i(t)}{N_i(t)}\right)\right] \geq 0.$$

Proof.

$$N_i(t)\phi_i(t)\frac{d}{dt}H\left(\frac{X_i(t)}{N_i(t)}\right) = N_i(t)\phi_i(t)H'\left(\frac{X_i(t)}{N_i(t)}\right)\frac{d}{dt}\left(\frac{X_i(t)}{N_i(t)}\right)$$

$$= \phi_i(t)H'\left(\frac{X_i(t)}{N_i(t)}\right)\left[\frac{d}{dt}X_i(t) - \frac{X_i(t)\frac{d}{dt}N_i(t)}{N_i(t)}\right]$$

$$= \phi_i(t)H'\left(\frac{X_i(t)}{N_i(t)}\right)\left[\sum_j a_{ij}(t)X_j(t) - \lambda_{per}X_i(t) - \frac{X_i(t)\left(\sum_j a_{ij}(t)N_j(t) - \lambda_{per}N_i(t)\right)}{N_i(t)}\right]$$

$$= \phi_i(t)H'\left(\frac{X_i(t)}{N_i(t)}\right)a_{ij}(t)N_j(t)\phi_i(t)\sum_j\left(\frac{X_j(t)}{N_j(t)} - \frac{X_i(t)}{N_i(t)}\right)$$

and

$$\phi_i(t)H\left(\frac{X_i(t)}{N_i(t)}\right)\frac{d}{dt}N_i(t) + N_i(t)H\left(\frac{X_i(t)}{N_i(t)}\right)\frac{d}{dt}\phi_i(t)$$

$$= \phi_i(t)H\left(\frac{X_i(t)}{N_i(t)}\right)\left(\sum_j a_{ij}(t)N_j(t) - \lambda_{per}N_i(t)\right) + N_i(t)H\left(\frac{X_i(t)}{N_i(t)}\right)\left(-\sum_j a_{ji}(t)\phi_j(t) + \lambda_{per}\phi_i(t)\right)$$

$$= \phi_i(t)H\left(\frac{X_i(t)}{N_i(t)}\right)\sum_j a_{ij}(t)N_j(t) - N_i(t)H\left(\frac{X_i(t)}{N_i(t)}\right)\sum_j a_{ji}(t)\phi_j(t).$$

Thus,

$$\frac{d}{dt}\sum_{i=1}^{d} H\left(\frac{X_i(t)}{N_i(t)}\right)N_i(t)\phi_i(t) = -D_H(X)(t).$$

Corollary 2.3. A unique solution (up to a multiplicative constant) exists for the Floquet eigenvalue problem denoted by Equations 2.1, 2.2 and 2.3.

Proof. Using the general relative entropy property with the convex function $H(s) = (s-1)^2$,

$\frac{d}{dt}\sum_{i=1}^{d} N_i(t)\phi_i(t)\left(\frac{X_i(t)}{N_i(t)} - 1\right)^2 = -\sum_{i,j} a_{ij}(t)N_j(t)\phi_i(t)\left(\frac{X_j(t)}{N_j(t)} - \frac{X_i(t)}{N_i(t)}\right)^2 \leq 0$

Thus $\sum_{i=1}^{d} N_i(t)\phi_i(t)\left(\frac{X_i(t)}{N_i(t)} - 1\right)^2$ is a positive, periodic and decreasing function, hence, it is constant and

$\sum_{i,j} a_{ij}(t)N_j(t)\phi_i(t)\left(\frac{X_j(t)}{N_j(t)} - \frac{X_i(t)}{N_i(t)}\right)^2 = 0.$

This is only possible when for all $i, j = 1, \ldots, d$,

$$\frac{X_j(t)}{N_j(t)} = \frac{X_i(t)}{N_i(t)} = c(t).$$

It can now be proven that in the case where $X_i(t) = c(t)N_i(t)$, $c(t)$ must be constant. Using Equations 2.1 and 2.2

$$\begin{aligned}\frac{d}{dt}X_i(t) &= N_i(t)\frac{d}{dt}c(t) + c(t)\frac{d}{dt}N_i(t)\\ &= N_i(t)\frac{d}{dt}c(t) + c(t)\left(\sum_j a_{ij}\, N_j(t) - \lambda_{per}N_i(t)\right)\\ &= N_i(t)\frac{d}{dt}c(t) + \left(\sum_j a_{ij}\, X_j(t) - \lambda_{per}X_i(t)\right)\\ &= N_i(t)\frac{d}{dt}c(t) + \frac{d}{dt}X_i(t).\end{aligned}$$

Then $N_i(t)\frac{d}{dt}c(t) = 0$. Since $N_i(t) > 0$, $\frac{d}{dt}c(t) = 0$. So $c(t)$ is constant as required.

Asymptotic behaviour

Here, the maximum entropy principle is used to prove exponential decay.

Proposition 2.4. If c and C are constants, such that $cN_i(0) \le X_i(0) \le CN_i(0)$, then it holds that

$$cN_i(t) \le X_i(t) \le CN_i(t)\ for\ any\ t > 0.$$

Furthermore, a constant $\alpha > 0$ such that

$$\sum_{i=1}^{d} N_i(t)\phi_i(t)\left(\frac{X_i(t)}{N_i(t)} - 1\right)^2 \le \sum_{i=1}^{d} N_i(0)\phi_i(0)\left(\frac{X_i(0)}{N_i(0)} - 1\right)^2 e^{-\alpha t}.$$

Proof. By applying the entropy principle to the convex function $H(s) = max(0, s - C)$, it may be shown that $\sum_{i=1}^{d} N_i(0)\phi_i(0)H\left(\frac{X_i(0)}{N_i(0)}\right) = 0$. However, as general relative entropy is nonnegative and decaying, it remains zero at all times.

$$\sum_{i=1}^{d} N_i(t)\phi_i(t)H\left(\frac{X_i(t)}{N_i(t)}\right) = 0\ for\ any\ t > 0.$$

Since $N_i(t), \phi_i(t) > 0$, $H\left(\frac{X_i(t)}{N_i(t)}\right) = 0$; that is $X_i(t) \le CN_i(t)$. Similarly, for the convex function $H(s) = max(0, c - s)$, it can be shown that $cN_i(t) \le X_i(t)$.

To prove the second claim, the entropy property may be applied to the convex function $H(s) = (s-1)^2$ to obtain

$$\frac{d}{dt}\sum_{i=1}^{d} N_i(t)\phi_i(t)\left(\frac{X_i(t)}{N_i(t)} - 1\right)^2 = -\sum_{i,j} a_{ij}(t)N_j(t)\phi_i(t)\left(\frac{X_j(t)}{N_j(t)} - \frac{X_i(t)}{N_i(t)}\right)^2$$

$$\leq -\alpha \sum_i \phi_i(t) N_i(t) \left(\frac{X_i(t)}{N_i(t)} - 1 \right)^2 ,$$

where **Lemma 2.5 (**below) and the Gronwall's inequality (see Appendix) are used.

Lemma 2.5. If $\phi(t), N(t) > 0, a_{ij}(t) > 0$ for all $i, j = 1, \dots, d, i \neq j$, then there is a constant $\alpha > 0$ such that the following inequality holds

$$\sum_{i,j=1}^{d} \phi_i(t) a_{ij}(t) N_j(t) \left(\frac{m_j(t)}{N_j(t)} - \frac{m_i(t)}{N_i(t)} \right)^2 \geq \alpha \sum_{i=1}^{d} \phi_i(t) N_i(t) \left(\frac{m_i(t)}{N_i(t)} \right)^2$$

for all m such that $\sum_{i=1}^{d} \phi_i(t) m_i(t) = 0.$

Proof. For the case $m(t) = 0$, the proof is trivial. So a case where case $m(t) \neq 0$ and is considered and normalized so that $\sum_{i=1}^{d} \phi_i(t) N_i(t) \left(\frac{m_i(t)}{N_i(t)} \right)^2 = 1$. This case is proven by contradiction. If no such α exists, a sequence $\left(m^k(t) \right)_{k \geq 1}$ can be constructed, with

$$\sum_{i,j=1}^{d} \phi_i(t) a_{ij}(t) N_j(t) \left(\frac{m_j^k(t)}{N_j(t)} - \frac{m_i^k(t)}{N_i(t)} \right)^2 \leq \frac{1}{k}$$

and

$$\sum_{i=1}^{d} \phi_i(t) N_i(t) \left(\frac{m_i^k(t)}{N_i(t)} \right)^2 = 1.$$

The compactness of $\left(m^k(t) \right)_{k \geq 1}$ follows from the Arzela-Ascoli theorem, so a convergent subsequence can be extracted still denoted by $\left(m^k(t) \right)_{k \geq 1}$ with

$\lim_{k\to\infty} m^k(t) = \bar{m}(t)$. Then passing to the limit gives $\sum_{i=1}^{d} \phi_i(t)N_i(t)\left(\frac{\overline{m_i}(t)}{N_i(t)}\right)^2 = 1$ and

$$\sum_{i,j=1}^{d} \phi_i(t)a_{ij}(t)N_j(t)\left(\frac{\overline{m_j}(t)}{N_j(t)} - \frac{\overline{m_i}(t)}{N_i(t)}\right)^2 = 0.$$

By the positivity of $\phi_i(t), N_i(t), a_{ij}(t)$, $\frac{\overline{m_i}(t)}{N_i(t)} = \frac{\overline{m_j}(t)}{N_j(t)} = \nu(t)$, for all $i, j = 1, \dots, d$. However since $0 = \sum_{i=1}^{d} \phi_i(t)N_i(t)\frac{\overline{m_i}(t)}{N_i(t)} = \nu(t)\sum_{i=1}^{d} \phi_i(t)N_i(t)$, it follows that $\nu(t) = 0$, which contradicts to $\sum_{i=1}^{d} \phi_i(t)N_i(t)\left(\frac{\bar{m}(t)}{N_i(t)}\right)^2 = 1$.

Application to an age-structured equation

Now a model of dynamics of population age-structured can be considered in which the coefficients are a periodic function of time. This is described by the following Von Forester-McKendrick partial differential equation

$$\begin{cases} \frac{\partial}{\partial t}n(t,x) + \frac{\partial}{\partial x}n(t,x) + d(t,x)n(t,x) = 0, \ \forall t \geq 0, x \geq 0 \\ n(t, x=0) = \int_0^\infty B(t,x)n(t,x)dx \\ n(t=0, x) = n^0(x), \end{cases} \tag{3.1}$$

where $n(t,x)$ is a population density of individuals of age $x > 0$ at a time $t \in (0,\infty)$ with $d(t,x)$ and $B(t,x)$ representing the death and birth rate of a population and being T-periodic, respectively. The boundary condition at $x = 0$ represents the number of newborns at time t and $n^0(x)$ is the initial age distribution of the population at time $t = 0$. It is assumed that $d \geq 0, B > 0, d, B \in W^{1,\infty}$ and

$$1 < \inf_{t\in(0,T)} \int_0^\infty B(.,x)e^{-\int_0^x d(.-x+y,y)dy}\,dx,$$

$$\sup_{t\in(0,T)} \int_0^\infty B(.,x)e^{-\int_0^x d(.-x+y,y)dy}\, dx < \infty.$$

Then Equation 3.1 can be written as an evolution equation

$$\begin{cases} \dfrac{\partial}{\partial t} n = An \\ n(0,x) = n^0(x) \end{cases}$$

with the operator $An = -\frac{\partial}{\partial x} n - dn$ is defined on the space

$$E = \left\{ n(t,x) \in \mathcal{D}'\big((0,\infty)\times(0,\infty)\big) \middle| n(t,0) = \int_0^\infty B(t,x)n(t,x)dx \right\}.$$

The long-run asymptotic behaviour of the population density, with a growth rate measured by the Floquet exponent λ_{per} using the Floquet eigenvalue problem can now be studied

$$\begin{cases} \frac{\partial}{\partial t} N(t,x) + \frac{\partial}{\partial x} N(t,x) + \big(\lambda_{per} + d(t,x)\big) N(t,x) = 0, \ \forall t \geq 0, x \geq 0 \\ N(t,x=0) = \int_0^\infty B(t,x)N(t,x)dx \\ N(t,x) > 0, T-\mathit{periodic}, \int_0^T \int_0^\infty N(t,x)dxdt = 1 \end{cases} \tag{3.2}$$

together with its adjoint eigenvalue problem

$$\begin{cases} -\frac{\partial}{\partial t}\phi(t,x) - \frac{\partial}{\partial x}\phi(t,x) + \big(\lambda_{per} + d(t,x)\big)\phi(t,x) = B(t,x)\phi(t,0), \\ \qquad \forall t \geq 0, x \geq 0 \\ \phi(t,x) > 0, T-\mathit{periodic}, \int_0^\infty N(t,x)\phi(t,x)dx = 1 \end{cases} \tag{3.3}$$

First, the existence and uniqueness of the following partial differential equation are considered

Theorem 3.1. If $\mu > 0$, then a unique solution $n \in C\big(R_+, L^1(R_+;\phi(.,x)dx)\big)$ to the below equation exists

$$\begin{cases} \frac{\partial}{\partial t}n(t,x) + \frac{\partial}{\partial x}n(t,x) + \big(\mu + d(t,x)\big)n(t,x) = 0, \ \forall t \geq 0, x \geq 0 \\ n(t, x=0) = \int_0^\infty B(t,x)n(t,x)dx \\ n(t=0,x) = n^0(x) \in L^1(R_+; \phi(0,x)dx). \end{cases}$$

Proof. The Banach-Fixed point theorem in the Banach space $X = C\big([0,T], L^1(R_+;dx)\big)$ endowed with the norm $\|n\|_X = sup_{t\in[0,T]}\|n(t,.)\|_{L^1(R_+)}$ and for a given $n^0 \in L^1(R_+;dx)$ is used to show that $n(t,x)$ is a fixed point of a contraction operator. The operator is defined as follows.

$$\begin{aligned} U: X &\to X \\ m &\mapsto n = U(m), \end{aligned}$$

where n is a solution of

$$\begin{cases} \frac{\partial}{\partial t}n(t,x) + \frac{\partial}{\partial x}n(t,x) + \big(\mu + d(t,x)\big)n(t,x) = 0, \\ n(t, x=0) = \int_0^\infty B(t,x)m(t,x)dx \\ n(t=0,x) = n^0(x). \end{cases}$$

If $m_1, m_2 \in X$ and $n_i = U(m_i), i = 1,2,$ then the difference $n = n_1 - n_2$ satisfies

$$\begin{cases} \frac{\partial}{\partial t}n(t,x) + \frac{\partial}{\partial x}n(t,x) + \big(\mu + d(t,x)\big)n(t,x) = 0, \\ n(t, x=0) = \int_0^\infty B(t,x)m(t,x)dx \\ n(t=0,x) = 0, \end{cases}$$

where $= m_1 - m_2$. It also holds that

$$\begin{cases} \dfrac{\partial}{\partial t}|n(t,x)| + \dfrac{\partial}{\partial x}|n(t,x)| + \big(\mu + d(t,x)\big)|n(t,x)| = 0, \\ |n(t, x=0)| = \left|\displaystyle\int_0^\infty B(t,x)m(t,x)dx\right| \\ |n(t=0,x)| = 0. \end{cases}$$

By the characteristics method,

$$n(t,x) = \begin{cases} 0,, x \geq t \\ n(t-x,0)e^{-\int_0^x (\mu+d)(t-x+y,y)dy}, x < t \end{cases}$$

Since d, B are positive and bounded, then there is a constant $M > 0$ such that

$$\left|B(t,x)e^{-\int_0^x d(t-x+y,y)dy}\right| \leq M.$$

Thus,

$$\|n(t,.)\|_{L^1(R_+)} = \int_0^t |n(t,x)|dx = \int_0^t |n(t-x,0)|e^{-\int_0^x (\mu+d)(t-x+y,y)dy}\, dx$$

$$= \int_0^t \left|\int_0^\infty B(t-x,z)m(t-x,z)dz\right| e^{-\int_0^x (\mu+d)(t-x+y,y)dy}\, dx$$

$$\leq M \int_0^t \|m(t,.)\|_{L^1(R_+)}\, dx = tM\|m(t,.)\|_{L^1(R_+)}.$$

Hence,

$$\|n\|_X = sup_{t\in[0,T]}\|n(t,.)\|_{L^1(R_+)} \leq sup_{t\in[0,T]}tM\|m(t,.)\|_{L^1(R_+)}$$
$$= TM\|m\|_X.$$

And it is proven that $U: X \to X$. By selecting T so that $M \leq \frac{1}{2}$, it follows that

$$\|U(m_1) - U(m_2)\|_X \leq \frac{1}{2}\|m_1 - m_2\|_X.$$

This means that U is a contraction in the Banach space X, which proves the existence of the fixed point. This process can be iterated on the intervals $[T, 2T]$, $[2T, 3T]$,… to build a solution in $C\big(R_+, L^1(R_+; dx)\big)$. Next, the density argument is used to complete the proof: Let $n^0 \in L^1(R_+; \phi(0,x)dx), \exists n_k^0 \in L^1(R_+; dx)$ such that $n_k^0 \to n^0$ in $L^1(R_+; \phi(.,x)dx)$, and $\widetilde{n_k}(t,x)$ be solution of

$$\begin{cases} \frac{\partial}{\partial t}\widetilde{n_k}(t,x) + \frac{\partial}{\partial x}\widetilde{n_k}(t,x) + \big(\mu + d(t,x)\big)\widetilde{n_k}(t,x) = 0 \\ \widetilde{n_k}(t, x=0) = \int_0^\infty B(t,x)\widetilde{n_k}(t,x)dx\,. \end{cases}$$

If $\tilde{n} = \widetilde{n_k} - \widetilde{n_p}$, then

$$\frac{\partial}{\partial t}\big(\tilde{n}(t,x)\phi(t,x)\big) + \frac{\partial}{\partial x}\big(\tilde{n}(t,x)\phi(t,x)\big) = -\phi(t,0)B(t,x)\tilde{n}(t,x)$$

$$\phi(t,0)\tilde{n}(t,0) = \phi(t,0)\int_0^\infty B(t,x)\tilde{n}(t,x)dx$$

And it also holds that

$$\frac{\partial}{\partial t}\big(|\tilde{n}(t,x)|\phi(t,x)\big) + \frac{\partial}{\partial x}\big(|\tilde{n}(t,x)|\phi(t,x)\big) = -\phi(t,0)B(t,x)|\tilde{n}(t,x)|$$

$$\phi(t,0)|\tilde{n}(t,0)| = \phi(t,0)\left|\int_0^\infty B(t,x)\tilde{n}(t,x)dx\right|.$$

Integrating with x gives

$$\frac{d}{dt}\int_0^\infty \big(|\tilde{n}(t,x)|\phi(t,x)\big)dx \le 0.$$

And finally,

$$\int_0^\infty \left|\widetilde{n_k} - \widetilde{n_p}\right|\phi(t,x)dx \le \int_0^\infty \left|n_k^0 - n_p^0\right|\phi(0,x)dx.$$

Thus, $\tilde{n}$ is a Cauchy sequence in a Banach space $C\big(R_+, L^1(R_+; \phi(.,x)dx)\big)$. So $\tilde{n}$ converges in the space to a solution in the distribution sense.

Corollary 3.2. With the assumptions on d and B as above, there is a unique $\lambda_{per} > 0$ and $N, \phi \in C\big(R_+, L^1(R_+; \phi(.,x)dx)\big)$ of the Floquet eigenvalue problem in Equation 3.2 and its adjoint eigenvalue problem in Equation 3.3.

Proof. If $\lambda_{per} = \mu > 0$, then $N(t,x) \in C\big(R_+, L^1(R_+; \phi(.,x)dx)\big)$ exists by **Theorem 3.1.** It satisfies

$$\frac{\partial}{\partial t}N(t,x) + \frac{\partial}{\partial x}N(t,x) + \Big(\lambda_{per} + d(t,x)\Big)N(t,x) = 0.$$

Similarly, its adjoint is given by

$$-\frac{\partial}{\partial t}\phi(t,x) - \frac{\partial}{\partial x}\phi(t,x) + \Big(\lambda_{per} + d(t,x)\Big)\phi(t,x) = B(t,x)\phi(t,0),$$

where $\phi(t,x) \in C\big(R_+, L^1(R_+; \phi(.,x)dx)\big)$. Moreover, the operator U is strictly positive in $C\big(R_+, L^1(R_+; dx)\big)$ and T-periodic as soon as d, B are. It is also compact since $sup\ \{\|U(n)\|_X; \|n\|_X \leq 1\}$ is uniformly bounded hence equicontinuous and compactness follow from the Arzela-Ascoli theorem.

Then by **Corollary 1.11** and **Corollary 1.14** with $\lambda_{\text{per}} = \mu$ such that $r(\mu) = 1$, the spectral radius of U and up to renormalization N, ϕ is unique. To end the proof, μ needs to be found such that $r(\mu) = 1$. Since r is decreasing function and vanishes at infinity and

$$r(0) \geq inf \int_0^\infty B(.,x)e^{-\int_0^x d(.-x+y,y)dy}\,dx > 1.$$

It follows that a unique λ_{per} exists such that $r\big(\lambda_{per}\big) = 1$.

Long run asymptotic: exponential decay

In this section, long-run asymptotic exponential decay will be proven.

Theorem 3.3. Under the assumptions for d and B above and an additional assumption that $\exists\alpha > 0$ such that $B(t,x) \geq \alpha\frac{\phi(t,x)}{\phi(t,0)}$, it follows that

$$\int_0^\infty \left| n(t,x)e^{-\lambda_{per}t} - \rho N(t,x) \right| \phi(t,x)dx \le e^{-\alpha t} \int_0^\infty |n^0(x) - \rho N(0,x)| \phi(0,x)dx,$$

where $\rho = \int_0^\infty n^0(x)\phi(0,x)dx$.

Proof. By taking $h(t,x) = n(t,x)e^{-\lambda_{per}t} - \rho N(t,x)$ and using Equations 3.1, 3.2 and 3.3

$$\frac{\partial}{\partial t}\big(h(t,x)\phi(t,x)\big) + \frac{\partial}{\partial x}\big(h(t,x)\phi(t,x)\big) = -\phi(t,0)B(t,x)h(t,x)$$

$$\phi(t,0)h(t,0) = \phi(t,0)\int_0^\infty B(t,x)h(t,x)dx.$$

By integrating with respect to x, it follows that

$$\frac{d}{dt}\int_0^\infty h(t,x)\phi(t,x)dx = -\phi(t,0)\int_0^\infty B(t,x)h(t,x)dx + h(t,0)\phi(t,0) = 0.$$

Then,

$$\begin{aligned}\int_0^\infty h(t,x)\phi(t,x)dx &= \int_0^\infty h(0,x)\phi(0,x)dx \\ &= \int_0^\infty \big(n^0(x) - \rho N(0,x)\big)\phi(0,x)dx \\ &= \int_0^\infty n^0(x)\phi(0,x)dx - \rho\int_0^\infty N(0,x)\phi(0,x)dx \\ &= \int_0^\infty n^0(x)\phi(0,x)dx - \rho = 0.\end{aligned}$$

And it also holds that,

$$\frac{\partial}{\partial t}\big(|h(t,x)|\phi(t,x)\big)+\frac{\partial}{\partial x}\big(|h(t,x)|\phi(t,x)\big)=-\phi(t,0)B(t,x)|h(t,x)|$$

$$\phi(t,0)|h(t,0)|=\phi(t,0)\left|\int_0^\infty B(t,x)h(t,x)dx\right|.$$

Now integrating with respect to x,

$\frac{\partial}{\partial t}\int_0^\infty |h(t,x)|\phi(t,x)\,dx=-\phi(t,0)\int_0^\infty B(t,x)|h(t,x)|dx+$
$|h(t,0)|\phi(t,0)$

$$\le -\phi(t,0)\int_0^\infty B(t,x)|h(t,x)|dx$$
$$+\left|\int_0^\infty [\phi(t,0)B(t,x)h(t,x)-\alpha\phi(t,x)h(t,x)]dx\right|$$
$$\le -\phi(t,0)\int_0^\infty B(t,x)|h(t,x)|dx$$
$$+\int_0^\infty \big(\phi(t,0)B(t,x)-\alpha\phi(t,x)\big)|h(t,x)|dx$$
$$\le -\alpha\int_0^\infty \phi(t,x)|h(t,x)|dx.$$

The proof is completed with Gronwall's inequality.

Long run asymptotic by the entropy method

Now long run asymptotic behaviour is proven by the entropy method.

Theorem 3.4.

1. For all convex function H and all $t>0$; it holds that

$$\frac{d}{dt}\int_0^\infty \phi(t,x)N(t,x)H\left(\frac{n(t,x)e^{-\lambda_{per}t}}{N(t,x)}\right)dx=-D_H(n)(t)\le 0$$

where

$$D_H(n)(t) = \phi(t,0)N(t,0)\left[\int_0^\infty H\left(\frac{n(t,x)e^{-\lambda_{per}t}}{N(t,x)}\right)d\mu_t - H\left(\int_0^\infty \frac{n(t,x)e^{-\lambda_{per}t}}{N(t,x)}\,d\mu_t(x)\right)\right].$$

2. For the probability measure $d\mu_t(x) = [B(t,x)N(t,x)/N(t,0)]dx$ and for all convex functions $H: R_+ \to R_+$; it holds that

$$\int_0^\infty \left[\int_0^\infty H\left(\frac{n(t,x)e^{-\lambda_{per}t}}{N(t,x)}\right)d\mu_t(x) - H\left(\int_0^\infty \frac{n(t,x)e^{-\lambda_{per}t}}{N(t,x)}\,d\mu_t(x)\right)\right]dt \leq K\int_0^\infty \phi(0,x)N(0,x)H\left(\frac{n^0(x)}{N(0,x)}\right)dx.$$

Proof. Using Equations 3.1 and 3.2,

$$\frac{\partial}{\partial t}\left(\frac{n(t,x)e^{-\lambda_{per}t}}{N(t,x)}\right) + \frac{\partial}{\partial x}\left(\frac{n(t,x)e^{-\lambda_{per}t}}{N(t,x)}\right) = 0.$$

Hence,

$$\frac{\partial}{\partial t}H\left(\frac{n(t,x)e^{-\lambda_{per}t}}{N(t,x)}\right) + \frac{\partial}{\partial x}H\left(\frac{n(t,x)e^{-\lambda_{per}t}}{N(t,x)}\right) = 0.$$

And finally, it holds that

$$\frac{\partial}{\partial t}\left[\phi(t,x)N(t,x)H\left(\frac{n(t,x)e^{-\lambda_{per}t}}{N(t,x)}\right)\right] + \frac{\partial}{\partial x}\left[\phi(t,x)N(t,x)H\left(\frac{n(t,x)e^{-\lambda_{per}t}}{N(t,x)}\right)\right]$$

$$= N(t,x)H\left(\frac{n(t,x)e^{-\lambda_{per}t}}{N(t,x)}\right)\left(\frac{\partial}{\partial t}\phi(t,x) + \frac{\partial}{\partial x}\phi(t,x)\right)$$

$$+\phi(t,x)H\left(\frac{n(t,x)e^{-\lambda_{per}t}}{N(t,x)}\right)\left(\frac{\partial}{\partial t}N(t,x)+\frac{\partial}{\partial x}N(t,x)\right)$$

$$+\phi(t,x)N(t,x)\left[\frac{\partial}{\partial t}H\left(\frac{n(t,x)e^{-\lambda_{per}t}}{N(t,x)}\right)+\frac{\partial}{\partial x}H\left(\frac{n(t,x)e^{-\lambda_{per}t}}{N(t,x)}\right)\right]$$

$$=-B(t,x)\phi(t,0)N(t,x)H\left(\frac{n(t,x)e^{-\lambda_{per}t}}{N(t,x)}\right).$$

Integrating with x and using the notation $d\mu_t(x)=[B(t,x)N(t,x)/N(t,0)]dx$, which is a probability measure

$$\frac{d}{dt}\int_0^\infty \phi(t,x)N(t,x)H\left(\frac{n(t,x)e^{-\lambda_{\text{per}}t}}{N(t,x)}\right)dx$$

$$=-\int_0^\infty \frac{\partial}{\partial x}\left[\phi(t,x)N(t,x)H\left(\frac{n(t,x)e^{-\lambda_{\text{per}}t}}{N(t,x)}\right)\right]dx$$

$$-\int_0^\infty B(t,x)\phi(t,0)N(t,x)H\left(\frac{n(t,x)e^{-\lambda_{\text{per}}t}}{N(t,x)}\right)dx$$

$$=\phi(t,0)N(t,0)H\left(\frac{n(t,0)e^{-\lambda_{per}t}}{N(t,0)}\right)$$

$$-\phi(t,0)N(t,0)\int_0^\infty H\left(\frac{n(t,x)e^{-\lambda_{per}t}}{N(t,x)}\right)d\mu_t(x)$$

$$=\phi(t,0)N(t,0)\left[H\left(\int_0^\infty \frac{n(t,x)e^{-\lambda_{per}t}}{N(t,x)}d\mu_t(x)\right)\right.$$

$$\left.-\int_0^\infty H\left(\frac{n(t,x)e^{-\lambda_{per}t}}{N(t,x)}\right)d\mu_t(x)\right].$$

The last quantity is negative because of Jensen's inequality. This shows that $\int_0^\infty \phi(t,x)N(t,x)H\left(\frac{n(t,x)e^{-\lambda_{per}t}}{N(t,x)}\right)dx$ is decaying and so the first inequality is found. By integrating again in t, the second inequality is obtained.

Theorem 3.5. Under the assumptions for $d\ and\ B$ above and $n^0 \in L^1(R_+, \phi(0,x)dx)$, it holds that

$$\int_0^\infty \left|n(t,x)e^{-\lambda_{per}t} - \rho N(t,x)\right|\phi(t,x)dx \to 0\ as\ t \to \infty,$$

where $\rho = \int_0^\infty n^0(x)\phi(0,x)dx$.

Proof. By setting $h(t,x) = n(t,x)e^{-\lambda_{per}t} - \rho N(t,x),\ h$ satisfies the equation

$$\begin{cases} \frac{\partial}{\partial t}h(t,x) + \frac{\partial}{\partial x}h(t,x) + \left(\lambda_{per} + d(t,x)\right)h(t,x) = 0,\ \forall t \geq 0, x \geq 0 \\ h(t,x=0) = \int_0^\infty B(t,x)h(t,x)dx \end{cases}$$

(3.4)

It also holds

$$\frac{\partial}{\partial t}\left(|h(t,x)|\phi(t,x)\right) + \frac{\partial}{\partial x}\left(|h(t,x)|\phi(t,x)\right) = -\phi(t,0)B(t,x)|h(t,x)|$$

$$\phi(t,0)|h(t,0)| = \phi(t,0)\left|\int_0^\infty B(t,x)h(t,x)dx\right|.$$

Now integrating with respect to x,

$$\frac{\partial}{\partial t}\int_0^\infty |h(t,x)|\phi(t,x)\,dx$$

$$= -\phi(t,0)\int_0^\infty B(t,x)|h(t,x)|dx + |h(t,0)|\phi(t,0)\ \leq 0.$$

This yields that $\int_0^\infty |h(t,x)|\phi(t,x)dx$ is decaying and it is positive, so it converges to some value $L \geq 0$. It remains to prove that $L = 0$.

Now the solutions $h_k(t,x) = h(t+k,x) \in C\big(R_+, L^1(R_+;\phi(.,x)dx)\big)$ to Equation 3.4 are defined. If H is positive convex, then **Theorem 3.4** shows that a quantity I_k defined by

$$I_k = \int_0^\infty \left[\int_0^\infty H\left(\frac{h_k(t,x)}{N(t,x)}\right) d\mu_t(x) - H\left(\int_0^\infty \frac{h_k(t,x)}{N(t,x)} d\mu_t(x)\right)\right] dt$$
$$= \int_k^\infty \left[\int_0^\infty H\left(\frac{h(t,x)}{N(t,x)}\right) d\mu_t(x) - H\left(\int_0^\infty \frac{h(t,x)}{N(t,x)} d\mu_t(x)\right)\right] dt$$

is bounded. As the integrand is positive and integrable, it can be deduced that $\lim_{k\to\infty} I_k = 0$. Moreover, $h_k(t,x)$ satisfies the equation

$$\begin{cases} \dfrac{\partial}{\partial t} h_k(t,x) + \dfrac{\partial}{\partial x} h_k(t,x) + \big(\lambda_{per} + d(t,x)\big) h_k(t,x) = 0, \ \forall t \geq 0, x \geq 0 \\ h_k(t, x=0) = \displaystyle\int_0^\infty B(t,x) h_k(t,x) dx \\ \displaystyle\int_0^\infty h_k(t,x)\phi(t,x) dx = 0. \end{cases}$$

Then $h_k(t,x) \in L^1(R_+;\phi(.,x)dx)$ is bounded up to a subsequence, $h_k \rightharpoonup g$ weakly. Passing to the limit in the definition of I_k and using the convexity in weak limits,

$$\int_0^\infty \int_0^\infty H\left(\frac{g(t,x)}{N(t,x)}\right) d\mu_t(x) dt \leq \lim_{k\to\infty} \int_0^\infty \int_0^\infty H\left(\frac{h_k(t,x)}{N(t,x)}\right) d\mu_t(x) dt$$

$$= \int_0^\infty H\left(\int_0^\infty \frac{g(t,x)}{N(t,x)} d\mu_t(x)\right) dt.$$

The last equality is valid since $\lim_{k\to\infty} I_k = 0$. But from Jensen's inequality, the reverse inequality is also found. Hence,

$$\int_0^\infty \int_0^\infty H\left(\frac{g(t,x)}{N(t,x)}\right) d\mu_t(x)dt = \int_0^\infty H\left(\int_0^\infty \frac{g(t,x)}{N(t,x)} d\mu_t(x)\right) dt.$$

This strictly convex equality for H shows that for almost all $t > 0$ on the support of μ_t,

$$\frac{g(t,x)}{N(t,x)} = C(t).$$

The limit in the weak sense gives

$$\begin{cases} \dfrac{\partial}{\partial t} g(t,x) + \dfrac{\partial}{\partial x} g(t,x) + \left(\lambda_{per} + d(t,x)\right) g(t,x) = 0, \ \forall t \geq 0, x \geq 0 \\ g(t, x = 0) = \displaystyle\int_0^\infty B(t,x) g(t,x) dx \end{cases}$$

and

$$\frac{\partial}{\partial t} \frac{g(t,x)}{N(t,x)} + \frac{\partial}{\partial x} \frac{g(t,x)}{N(t,x)} = 0$$

Hence $\frac{g(t,x)}{N(t,x)} = C(t)$ and as a result

$$0 = \int_0^\infty g(t,x)\phi(t,x)dx = C(t) \int_0^\infty N(t,x)\phi(t,x)dx = C(t).$$

It can be concluded that $L = 0$ since $L = \int_0^\infty |g(t,x)| \phi(t,x)dx$.

Here, the following **Lemma 3.6** (Perthame, 2007, p. 100) was used.

Lemma 3.6. Any function $u = g/N$ satisfies

$$\frac{g}{N}\left(t, \Gamma(x)\right) = \frac{g}{N}(t,x) \ \forall t > 0, x \geq 0$$

and the fact that

$$\frac{\partial}{\partial t}\left(\frac{g(t,x)}{N(t,x)}\right) + \frac{\partial}{\partial x}\left(\frac{g(t,x)}{N(t,x)}\right) = 0$$

is constant.

Discussion and Conclusion

An age-structured model with both death and birth rates depending only on age (not varying in time) was analyzed for the existence of long-run behaviour. This model was based on the general relative entropy method in Perthame (2007).

In this work, an age-structured model with both death and birth rates of a population that depend on age and time, and that is periodic over time has been analyzed. Floquet theory was applied to Banach space to prove the existence and uniqueness of the solution of this age-structured equation. In addition, the general relative entropy method (Perthame, 2007) has been used to derive the asymptotic exponential decay of the solution for this setting.

The exponential rate of convergence guarantees that the solution reaches the steady-state fast enough to be observed in practice. The exponential decay rate is known in the case of non-constant coefficients (Gwiazda & Perthame, 2006). While in our case, the exponential decay holds for a wider class of data.

The existence and uniqueness of the solution for the Floquet eigenvalue problem for the periodic operator on Banach space have been proven, so as long as the models can be written as a partial differential evolution equation. It is now tempting to apply the Floquet theory on Banach space to more advanced models such as age-structured models with migration, growth-fragmentation equations or cell division equations (Mischler & Scher, 2016).

The aim of this work was twofold. On one hand, the existence and uniqueness of the solution of the Floquet eigenvalue problem on Banach space have been proven. On the other hand, the existence and uniqueness of the solution of the age-structured equation with positive and periodic coefficients have been proven. Moreover, long-run asymptotic exponential decay of the solution of the age-structured equation has been derived.

Appendix

Lemma (Gronwall's inequality). If $u \in C^1([0,T])$ satisfying $\frac{d}{dt}u(t) \leq \alpha u(t), for\ all\ t \in [0,T]$ where α is constant, then $u(t) \leq u(0)\mathrm{e}^{\alpha t}$.

Definition. If T be an operator on a Banach space, the spectrum of T is $\sigma(T) = \{\lambda \in C: (\lambda I - T)^{-1}$ does not exist$\}$. Thus, the spectral radius of T is $r(T) = sup\{\,|\lambda|: \lambda \in \sigma(T)\}$.

Theorem. The spectrum of a bounded linear operator coincides with the spectrum of its adjoint; that is, $\sigma(T) = \sigma(T^*)$. In particular, $r(T) = r(T^*)$.

Theorem (Perron-Frobenius) (Perthame, 2007, p 160). If A is a positive, irreducible matrix, $d \times d$; then the spectral radius $r(A)$ of A is a positive simple eigenvalue of A associated with a positive eigenvector.

Definition. A cone K in a real Banach space $(X, \|.\|)$ is a closed set of X if it satisfies

1. $0 \in K$
2. $x, y \in K$, then $\lambda x + \mu y \in K$, $\forall \lambda, \mu \geq 0$
3. $x \in K$ and $-x \in K$, then $x = 0$

On a real Banach space $(X, \|.\|)$ the order on a cone K is defined by

$$(x \geq y \Leftrightarrow x - y \in K) \text{ and } \big(x > y \Leftrightarrow x - y \in Int(K)\big)$$

A cone K is reproducible if $\forall x \in X, \exists y, z \in K, x = y - z.$

A cone K is normal if $0 \leq x \leq y \Rightarrow \|x\| \leq \|y\|$.

A dual cone of K is $K^* = \{y \in X^*, \forall x \in K, \langle y, x \rangle \geq 0\}$.

Theorem (Krein-Rutman) (Perthame, 2007, p. 175). If $(X, \|.\|)$ is a real Banach space, $K \subset X$ a reproducible and normal and T linear, compact and strictly positive (on K) operator. Then the spectral radius $r(T)$ of T is a positive simple eigenvalue of T associated with a positive eigenvector. In addition, if $Int(K^*)$ is non-empty, then $r(T)$ is also a positive simple eigenvalue of the adjoint operator T^* associated with a positive eigenvector.

Theorem (Banach-Fixed Point). If (X, d) is a complete metric space and $f: X \to X$ is a contraction; that is, $k \in [0,1)$ exists such that $for\ any\ x, y \in X$,

$$d\big(f(x), f(y)\big) \leq kd(x, y).$$

Then there exists a unique fixed point for f.

Theorem (Arzela-Ascoli). If (X, d) is a compact metric space. A subset $\mathcal{F}$ of $C(X)$ is relatively compact if and only if $\mathcal{F}$ is bounded and equicontinuous.

Corollary. If (X, d) is a compact metric space and $(f_n) \subset C(X)$ is a bounded sequence and equicontinuous in $C(X)$, then (f_n) has a uniformly convergent subsequence.

Acknowledgements

The author would like to acknowledge the reviewers of this work for their helpful comments and suggestions.

Author Biography

Meas Len graduated with a PhD in mathematics at Côte D'Azur University in 2017 and a Masters in Mathematics from Paris Dauphine University in 2014, both in France and a Masters in Mathematics from the Royal University of Phnom Penh in 2010 in Cambodia. Currently, he works at the Department of Mathematics of the Royal University of Phnom Penh.

References

Anderson, A. M. & May., A. M. (1991). Infectious Diseases of Humans. *Dynamics and Control, Oxford University Press, Oxford*.

Brown, B. M. Easthem, M. S. P. & Schmidt, K.M. (2013). *Periodic Differential Operators* (pp. 1-26). Springer.

Clairambault, J.,Michel, P. &Perthame, B. (2016). Circadian rhythm and tumour growth. *C.R.A Cad. Sci. Paris, Ser I 342*, 17-22.

Gwiazda, P. & Perthame, B. (2006). Invariants and exponential rate of convergence to steady-state in renewal equation. *Markov Process and Related Fields*, 413-424.

Iannelli, M. & Milner, F. (2017). *The Basic Approach to Age-Structured Population Dynamics* (pp. 50-51). Springer.

McKenDrick, A. G. (1926). Applications of mathematics to medical problems. *Proc. Edinb. Math. Soc. 44*, 98-130.

Michel, P. Mischler, S. & Perthame, B. (2004). General entropy equations for structured population models and scattering. *C. R. Math. Acad. Sci. Paris 338(9)*, 697-702.

Michel, P. Mischler, S. & Perthame, B. (2005). General relative entropy inequality: an illustration on growth models. *J. Math. Pures Appli. (9),84(9)*, 1235-1260.

Mischler, S. Scher. J. (2016). Spectral analysis of semigroups and growth-fragmentation equations. *Annalues de L'Institut Henri Poincaré C, Analyse non linéaire, Volume 33*, 849-898.

Perthame. B. (2007). *Transport equations in biology.* Frontiers in Mathematics, Birkhäuser Verlag, Basel.

von Foerster, H. (1959). Some remarks on changing population. *The Kinetics of Cellular Proliferation, Grune and Stratton, New York*, 282-407.